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TORSIONAL OSCILLATIONS OF A LAYER BONDED TO AN ELASTIC HALF-SPACE†

L. M. KEER,‡ H. H. JABALI§ and K. CHANTARAMUNGKORN

Department of Civil Engineering, The Technological Institute, Northwestern University, Evanston, Illinois 60201

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Abstract—The problem of a layer bonded to an elastic half-space, where the layer is driven by torsional oscillations of a bonded rigid circular disk, is solved by means of integral transform techniques. Using a standard technique, the problem is reduced to a Fredholm integral equation of the second kind, the kernel of which involves the calculation of principal value integrals. Dynamic stiffnesses are developed for a range of layer thicknesses, material properties, and frequencies.

INTRODUCTION

There has been recent interest in problems concerning the dynamic response of foundations with the goal of improving the dynamic models for the interaction between the soil and the structure. Recently, a considerable effort has been made to solve problems involving the response of bodies on an elastic half-space, and many two and three dimensional solutions to these problems in dynamic elasticity have been developed.

Of important practical interest is the response of a structure on a layered foundation and for such a study the dynamic stiffness of the foundation must be investigated. There are relatively few investigations that have determined the stiffness of an elastic layer. The most successful results have been obtained for the case in which the excitation is assumed to be axially symmetric torsion, since this is the simplest case. The problem was first solved by Collins[1] who considered the forced torsional oscillations of an elastic half-space and an elastic stratum. He reduced the problem to an an integral equation which could be solved by iteration for low frequencies and relatively large stratum thicknesses. Williams[2] solved the same problem by Green's function techniques and also pointed out a minor error in Collins' paper.

A simpler analysis was given by Gladwell[3] who formulated the problem by means of integral transform techniques to obtain an integral equation which he solved by iteration. Gladwell considered separately the cases when the bottom surface of the layer is fixed and when it is free. An approximation scheme was developed by Awojobi[4, 5] to obtain a reduction in the difficulty required to solve the integral equations for axially symmetric torsion. While it is true that the computer effort is diminished by using the Awojobi technique, it seems more essential to present the results of solutions which have a known degree

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‡ During 1972–73, Guggenheim Fellow, Department of Mathematics, University of Glasgow.

§ Present address: Sargent and Lundy Consulting Engineers, 140 S. Dearborn, Chicago, Illinois 60603.

of accuracy in order to provide a means by which an approximate solution can be compared. The results given in [4] and [5] will be compared to those derived by using the integral equation of Gladwell in a separate paper.

The purpose of the present analysis is to consider the problem of a layer which is bonded to an elastic half-space and which is excited by torsional oscillations arising from a bonded rigid circular disk. Integral transform techniques will be used to formulate the problem in a manner similar to those used by Gladwell. The problem will be reduced to a Fredholm integral equation of the second kind which can be solved by numerical techniques. Dynamic stiffnesses will be developed for a relatively large parameter range of layer thicknesses and frequencies.

FORMULATION OF PROBLEM

The axially symmetric problem of the torsional oscillations of a rigid circular disk of radius a, attached to an elastic layer bonded to an elastic half-space, is considered here. The thickness of the layer is h, and the geometry and coordinate system for the physical situation are shown in Fig. 1. The shear moduli and mass density of the layer and foundation are denoted by μ_1, ρ_1 and μ_2, ρ_2 respectively.

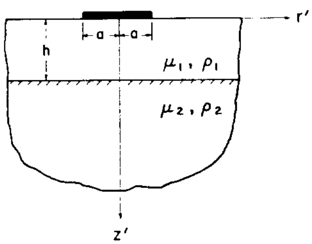

Fig. 1. Geometry and coordinate system.

In cylindrical coordinates (r', θ, z'), the equations of motion are

$$\mu_1 \frac{\partial}{\partial r'}\left[\frac{1}{r'}\frac{\partial}{\partial r'}(r'v_1')\right] + \mu_1 \frac{\partial^2 v_1'}{\partial z'^2} = \rho_1 \frac{\partial^2 v_1'}{\partial t^2} \tag{1}$$

and

$$\mu_2 \frac{\partial}{\partial r'}\left[\frac{1}{r'}\frac{\partial}{\partial r'}(r'v_2')\right] + \mu_2 \frac{\partial^2 v_2'}{\partial z'^2} = \rho_2 \frac{\partial^2 v_2'}{\partial t^2}, \tag{2}$$

where $v_1'(r', z', t)$ and $v_2'(r', z', t)$ are the displacements of the layer and foundation, respectively, and are in the direction of θ. If the dimensionless variables

$$r = r'/a, \qquad z = z'/a, \qquad d = h/a \tag{3}$$

are introduced, and harmonic motion of the form

$$v' = av(r, z)e^{i\omega t} \tag{4}$$

is assumed, where ω is the circular frequency, then solutions to equations (1) and (2) are sought to satisfy the boundary and continuity conditions, which are

$$v_1(r, 0) = \phi r \qquad (0 \le r < 1), \tag{5}$$

$$\sigma_{z\theta}{}^1(r, 0) = 0 \qquad (1 < r), \tag{6}$$

$$v_1(r, d) = v_2(r, d) \qquad (0 \leq r), \tag{7}$$

and

$$\sigma_{z\theta}{}^1(r, d) = \sigma_{z\theta}{}^2(r, d) \qquad (0 \leq r), \tag{8}$$

where ϕ is the angle of twist at maximum amplitude.

The displacement solutions to equations (1) and (2) can be obtained in the form of Hankel transforms as follows:

$$v_1(r, z) = \int_0^\infty [\cosh \beta_1(d - z) + (\delta\beta_2/\beta_1)\sinh \beta_1(d - z)] \frac{D(\xi)J_1(\xi r)}{\beta_1 f(\xi)} \, d\xi \tag{9}$$

and

$$v_2(r, z) = \int_0^\infty [e^{-\beta_2(z-d)}] \frac{D(\xi)J_1(\xi r)}{\beta_1 f(\xi)} \, d\xi, \tag{10}$$

where

$$f(\xi) = \sinh \beta_1 d + (\delta\beta_2/\beta_1)\cosh \beta_1 d, \tag{11}$$

$$\delta = \mu_2/\mu_1, \tag{12}$$

$$\beta_i^2 = \xi^2 - k_i^2, \qquad i = 1, 2 \tag{13}$$

and

$$k_i = a\omega(\rho_i/\mu_i)^{1/2}, \qquad i = 1, 2. \tag{14}$$

The corresponding stress components are easily obtained as

$$\sigma_{z\theta}{}^1(r, z) = -\mu_1 \int_0^\infty [\sinh \beta_1(d - z) + (\delta\beta_2/\beta_1)\cosh \beta_1(d - z)] \frac{D(\xi)J_1(\xi r)}{f(\xi)} \, d\xi \tag{15}$$

and

$$\sigma_{z\theta}{}^2(r, z) = -\mu_2 \int_0^\infty [\beta_2 e^{-\beta_2(z-d)}] \frac{D(\xi)J_1(\xi r)}{\beta_1 f(\xi)} \, d\xi. \tag{16}$$

In view of equations (9), (10), (15) and (16) it is verified that the continuity conditions, equations (7) and (8), are automatically satisfied. The remaining mixed boundary conditions, equations (5) and (6), are written in the following form as dual integral equations:

$$\int_0^\infty \frac{g(\xi)}{\beta_1 f(\xi)} D(\xi)J_1(\xi r) \, d\xi = \phi r \qquad (0 \leq r < 1), \tag{17}$$

$$\int_0^\infty D(\xi)J_1(\xi r) \, d\xi = 0 \qquad (1 < r), \tag{18}$$

where

$$g(\xi) = \cosh \beta_1 d + (\delta\beta_2/\beta_1)\sinh \beta_1 d. \tag{19}$$

The results as they are now written do not constitute a solution that has outgoing waves. To ensure that the solution will produce outgoing waves it is necessary to add an additional term to the displacements that will correspond to free waves. This procedure is outlined in the paper by Gladwell. The term which must be added to equation (17) is of the form

$$-\pi i \sum_{n=1}^{m} \frac{g(\xi_n)}{\beta_1(\xi_n)f'(\xi_n)} D(\xi_n)J_1(\xi_n r), \tag{20}$$

where ξ_n are the roots of $f(\xi)$, m is the number of roots that enters into the solution of the equation $f(\xi_n) = 0$, $k_2 < \xi_n < k_1$, and $f'(\xi_n)$ is $\mathrm{d}f/\mathrm{d}\xi$ evaluated at $\xi = \xi_n$. Thus equations (17) and (18) are modified to become

$$\int_0^\infty \frac{g(\xi)}{\beta_1 f(\xi)} D(\xi)J_1(\xi r)\,\mathrm{d}\xi - \pi i \sum_{n=1}^{m} \xi_n^{-1} F_n D(\xi_n)J_1(\xi_n r) = \phi r \qquad (0 \le r < 1), \tag{21}$$

$$\int_0^\infty D(\xi)J_1(\xi r)\,\mathrm{d}\xi = 0 \qquad (1 < r), \tag{22}$$

where

$$F_n = \frac{\gamma_{2n}\gamma_{1n}\cos(\gamma_{1n}d) + \delta\gamma_{2n}^2\sin(\gamma_{1n}d)}{\gamma_{2n}(1 + \delta\gamma_{2n}d)\sin(\gamma_{1n}d) + \gamma_{1n}(\delta + \gamma_{2n}d)\cos(\gamma_{1n}d)}, \tag{23}$$

$$\gamma_{1n} = (k_1^2 - \xi_n^2)^{1/2} \tag{24}$$

and

$$\gamma_{2n} = (\xi_n^2 - k_2^2)^{1/2}. \tag{25}$$

REDUCTION TO FREDHOLM INTEGRAL EQUATION

Equation (22) is automatically satisfied by assuming (see, e.g.[3]) the solution form

$$D(\xi) = (2\xi/\pi)\int_0^1 \theta(\eta)\sin(\xi\eta)\,\mathrm{d}\eta, \tag{26}$$

where the unknown auxiliary function $\theta(\eta)$ in equation (26) is to be determined from equation (21). The substitution of equation (26) into equation (21) and the use of an elementary identity[6] leads to the Abel integral equation

$$\frac{2}{\pi}\int_0^r \frac{\eta\theta(\eta)\,\mathrm{d}\eta}{r(r^2 - \eta^2)^{1/2}} = G(r) \qquad (0 \le r < 1), \tag{27}$$

where

$$G(r) = \phi r + (1/\pi)\int_0^1 \theta(\eta)\left[2\pi i \sum_{n=1}^{m} F_n \sin(\xi_n\eta)J_1(\xi_n r)\right]\mathrm{d}\eta - (2/\pi)\int_0^\infty\int_0^1 H(\xi)\theta(\eta)J_1(\xi r)\sin(\xi\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi \tag{28}$$

and

$$H(\xi) = g(\xi)/[\beta_1 f_1(\xi)] - 1. \tag{29}$$

The solution of equation (27) is elementary and leads to the inhomogeneous Fredholm integral equation

$$\theta(x) + (1/\pi)\int_0^1 M^*(x, \xi)\theta(\xi)\,\mathrm{d}\xi = 2\phi x \qquad (0 \le x < 1), \tag{30}$$

where the kernel $M^*(x, \xi)$ is given by

$$M^*(x, \xi) = M(x, \xi) - 2\pi i \sum_{n=1}^{m} F_n \sin(\xi_n x)\sin(\xi_n \xi) \tag{31}$$

and

$$M(x, \xi) = 2 \int_0^\infty H(\eta)\sin(x\eta)\sin(\xi\eta)\, \mathrm{d}\eta. \tag{32}$$

By introducing

$$s = \eta/k_1, \qquad \tau = k_1 d, \qquad c = k_2/k_1, \tag{33}$$

equation (32) becomes

$$M(x, \xi) = 2k_1 \int_0^\infty H(k_1 s)\sin(k_1 x s)\sin(k_1 \xi s)\, \mathrm{d}s, \tag{34}$$

where

$$H(k_1 s) = (s/p)\left[\frac{1 + \delta(q/p)\tanh(\tau p)}{\tanh(\tau p) + \delta(q/p)}\right] - 1, \tag{35}$$

$$p = (s^2 - 1)^{1/2} \tag{36}$$

and

$$q = (s^2 - c^2)^{1/2}. \tag{37}$$

It is observed that only β_2 will provide a branch point for the integrand in equation (30). Furthermore, the equation $f(\xi_n) = 0$ has no relevant solution unless k_1 is larger than k_2, that is for the case of a foundation that is stiffer than the layer. It is also observed that the real roots of $f(\xi_n) = 0$ occur only when $k_1 > \xi_n > k_2$. It is therefore convenient to rewrite equation (31) for the cases of (a) stiffer foundation medium and (b) stiffer layer medium.

(a) Stiffer foundation

When the foundation is stiffer than the layer then $k_1 > \xi_n > k_2$. In this case equation (34) may be written as

$$\begin{aligned} M(x, \xi) &= k_1 \int_0^c (s/\bar{p})\left[\frac{1 + i\delta(\bar{q}/\bar{p})\tan(\tau\bar{p})}{i\delta(\bar{q}/\bar{p}) - \tan(\tau\bar{p})}\right]\Psi(k_1, s)\, \mathrm{d}s \\ &+ k_1 \int_c^1 (s/\bar{p})\left[\frac{1 + \delta(q/\bar{p})\tan(\tau\bar{p})}{(\delta q/\bar{p}) + \tan(\tau\bar{p})}\right]\Psi(k_1, s)\, \mathrm{d}s \\ &- k_1 \int_0^1 (s/\bar{p})\chi(k_1, s)\, \mathrm{d}s \\ &+ k_1 \int_1^\infty (s/p)\left[\frac{1 + \delta(q/p)\tanh(\tau p)}{\delta(q/p) + \tanh(\tau p)} - 1\right]\Psi(k_1, s)\, \mathrm{d}s, \end{aligned} \tag{38}$$

where

$$\Psi(k_1, s) = \cos[k_1(x - \xi)s] - \cos[k_1(x + \xi)s], \tag{39}$$

$$\chi(k_1, s) = \sin[k_1 |x - \xi| s] - \sin[k_1(x + \xi)s], \tag{40}$$

$$\bar{p} = (1 - s^2)^{1/2} \tag{41}$$

and

$$\bar{q} = (c^2 - s^2)^{1/2}. \tag{42}$$

The kernel M^*, given by equation (31), is obtained by adding the contribution from the poles to M, given by equation (38). The details of the development of equation (38) have been omitted since they follow the same general procedure developed by Gladwell[3] for the case of layer on a rigid foundation.

(*b*) *Stiffer layer*

The stiffer layer corresponds to the case where $k_2 > k_1$. Here there will be no contribution from the poles in the kernel M^* and equation (31) may be written as

$$\begin{aligned} M^*(x, \xi) = {} & k_1 \int_0^1 (s/\bar{p}) \left[\frac{1 + i\,\delta(\bar{q}/\bar{p})\tan(\tau\bar{p})}{i\,\delta(\bar{q}/\bar{p}) - \tan(\tau\bar{p})} \right] \Psi(k_1, s)\,\mathrm{d}s \\ & + k_1 \int_1^c (s/p) \left[\frac{1 + i\,\delta(\bar{q}/p)\tanh(\tau p)}{i\,\delta(\bar{q}/p) + \tanh(\tau p)} \right] \Psi(k_1, s)\,\mathrm{d}s \\ & - k_1 \int_0^1 (s/\bar{p})\chi(k_1, s)\,\mathrm{d}s - k_1 \int_1^c (s/p)\Psi(k_1, s)\,\mathrm{d}s \\ & + k_1 \int_c^\infty (s/p) \left[\frac{1 + \delta(q/p)\tanh(\tau p)}{\delta(q/p) + \tanh(\tau p)} - 1 \right] \Psi(k_1, s)\,\mathrm{d}s. \end{aligned} \tag{43}$$

(*c*) *Limiting cases*

The solution for a *rigid foundation* can be obtained from equations (31) and (38) by taking the limiting values of M and F_n as δ approaches infinity. The result is

$$\begin{aligned} M^*(x, \xi) = {} & k_1 \int_0^1 (s/\bar{p})\tan(\tau\bar{p})\Psi(k_1, s)\,\mathrm{d}s - k_1 \int_0^1 (s/\bar{p})\chi(k_1, s)\,\mathrm{d}s \\ & + k_1 \int_1^\infty (s/p)[\tanh(\tau p) - 1]\Psi(k_1, s)\,\mathrm{d}s \\ & - (2\pi i/d) \sum_{n=1}^{m} \sin(\xi_n x)\sin(\xi_n \xi), \end{aligned} \tag{44}$$

where ξ_n are the roots of the equation

$$\cos[(k_1{}^2 - \xi_n{}^2)^{1/2}\, d] = 0. \tag{45}$$

The *free boundary* solution corresponds to the case when the foundation has zero stiffness with respect to the layer. The solution can be obtained by taking the limiting value of M^*

in equation (43) as δ approaches zero. However, the contribution of a simple pole at $\xi = k_1$ must be added to the imaginary part of the kernel thus giving

$$\begin{aligned} M^*(x, \xi) = & -k_1 \int_0^1 (s/\bar{p})\cot(\tau\bar{p})\Psi(k_1, s)\,\mathrm{d}s \\ & - k_1 \int_0^1 (s/\bar{p})\chi(k_1, s)\,\mathrm{d}s \\ & + k_1 \int_1^\infty (s/p)[\coth(\tau p) - 1]\Psi(k_1, s)\,\mathrm{d}s \\ & - (\pi i/d)\sin(k_1 x)\sin(k_1\xi) \\ & - (2\pi i/d)\sum_{n=1}^{m} \sin(\xi_n x)\sin(\xi_n \xi), \end{aligned} \tag{46}$$

where ξ_n are the roots of the equation

$$\sin[(\xi_n{}^2 - k_1{}^2)^{1/2}\, d] = 0. \tag{47}$$

Equations (44) and (46) are in agreement with Gladwell[3].

The *static solution* corresponds to $k_1 = k_2 = 0$ and for this case the kernel becomes

$$M^*(x, \xi) = 2 \int_0^\infty [1 + \alpha e^{2ds}]^{-1}\{\cos[(x + \xi)s] - \cos[(x - \xi)s]\}\,\mathrm{d}s \tag{48}$$

where

$$\alpha = (\delta + 1)/(\delta - 1). \tag{49}$$

The *half-space solution* is obtained when $\delta = 1$. The kernel reduces to

$$M^*(x, \xi) = -k_1 \int_0^1 (s/\bar{p})\chi(k_1, s)\,\mathrm{d}s - ik_1 \int_0^1 (s/\bar{p})\Psi(k_1, s)\,\mathrm{d}s, \tag{50}$$

in agreement with Robertson[7].

(d) Physical quantities

The objective of this paper is to obtain the dynamic stiffness for the system shown in Fig. 1. The important physical quantity to calculate therefore is the applied torque, which is given by

$$T = -2\pi a^3 \int_0^1 r^2 \sigma_{z\theta}{}^1(r, 0)\,\mathrm{d}r. \tag{51}$$

Equation (51) can be written in terms of the unknown function by the use of equation (15), evaluated at $z = 0$, equation (26) and certain identities[6] as

$$T = 8\mu_1 a^3 \int_0^1 t\theta(t)\,\mathrm{d}t. \tag{52}$$

For the static problem of an elastic half-space, equation (52) becomes

$$T = 16\mu_1 a^3 \phi/3. \tag{53}$$

NUMERICAL SOLUTION

By separating the real and imaginary parts, the Fredholm integral equation (30) is written as a set of two integral equations

$$\theta_r(x) + (1/\pi)\int_0^1 [M_r(x, \xi)\theta_r(\xi) - M_i(x, \xi)\theta_i(\xi)]\,d\xi = 2\phi x \tag{54}$$

and

$$\theta_i(x) + (1/\pi)\int_0^1 [M_r(x, \xi)\theta_i(\xi) + M_i(x, \xi)\theta_r(\xi)]\,d\xi = 0, \tag{55}$$

where θ_r, M_r and θ_i, M_i are the real and imaginary parts of the unknown function $\theta(x)$ and the complex kernel $M^*(x, \xi)$, respectively. Numerical solutions of equations (54) and (55) can be obtained by the standard technique of approximating the integrals by a sum over discrete values of ξ and thereby obtaining a system of linear inhomogeneous equations in the unknown $\theta_r(\xi_j)$ and $\theta_i(\xi_j)$.

When the parameters δ, d and k_1 are specified,† equations (54) and (55) yield a set of linear equations which can be solved simultaneously for $\theta_r(\xi_j)$ and $\theta_i(\xi_j)$. The integrals which form the real and imaginary parts of the kernel were evaluated by means of Simpson's rule. However, a special technique must be used for the evaluation of those integrals which are defined in the sense of their principal value. These integrals appear in equations (38), (44) and (46) and cannot be evaluated through standard numerical procedures. The evaluation of the principal value integrals was accomplished by normalizing the integrals and assuming their integrands to be antisymmetric about their singular points over a sufficiently small chosen interval. Therefore, the magnitude of the integrals in this small interval can be considered as negligible when compared to the contribution of the integrals from the entire interval. For the problems considered here an interval size of 0·002 was used to contain the singular point, and the integrals evaluated over the entire interval by this procedure yielded their values to within 0·2 per cent accuracy. The accuracy of the computation can be improved by decreasing the length of the neglected interval, but the interval of 0·002 is considered to be consistent with the accuracy of the remaining calculations for all cases considered here.

Numerical results were obtained for the cases when $\delta = 0{\cdot}1, 0{\cdot}2, 0{\cdot}5, 2, 5$ and 10. In each case values of $d = 1, 2, 5$ and 10 were used with various values of k_1, ranging from 0 to 1·5, with a 0·1 increment. The case when $\delta < 1{\cdot}0$ represents a layer which is stiffer than the foundation, while $\delta > 1{\cdot}0$ represents a stiffer foundation. The results are given in terms of curves shown in Figs. 2–7. The "real part" (*Re*) of the applied torque refers to the difference between the real part of the multiplier of $16\mu_1 a^3\phi/3$ in equation (52) and that of the same multiplier for the static, half-space solution given by equation (53). The "imaginary part" (*Im*) refers to the imaginary part of the same multiplier of $16\mu_1 a^3\phi/3$ in equation (53). It is noted that the imaginary part of the applied torque is always positive.

Physically the results are very interesting. It is observed from Figs. 2–7 that the imaginary part of the torque always has a small but nonzero value provided that k_1 is greater than zero. This is a distinctly different result from that of Gladwell[3] for a layer on a rigid foundation. For that case there is for each value of d a specific frequency below which the imaginary part of the torque vanishes. Therefore for that case it is possible to have a situation in which

† For convenience $\rho_1 = \rho_2$ in all calculations. If this were not the case k_2 would also have to be specified.

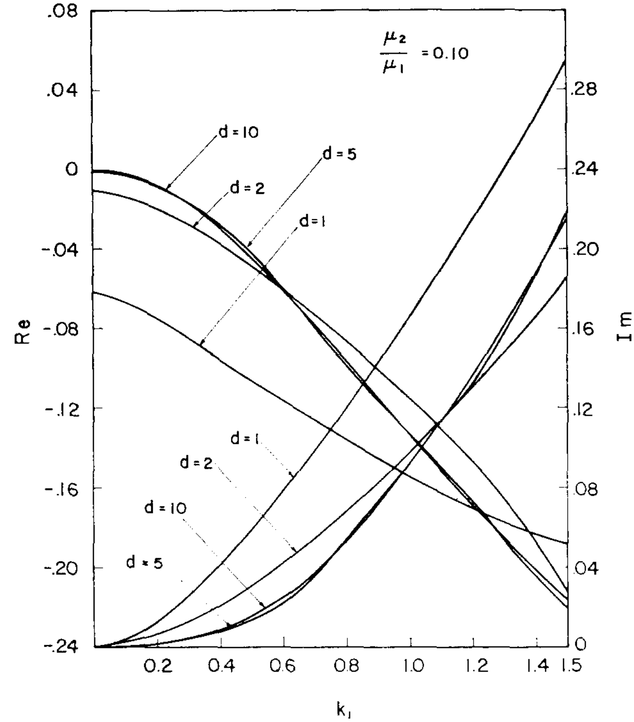

Fig. 2. Dynamic stiffnesses for $\delta = 0{\cdot}10$.

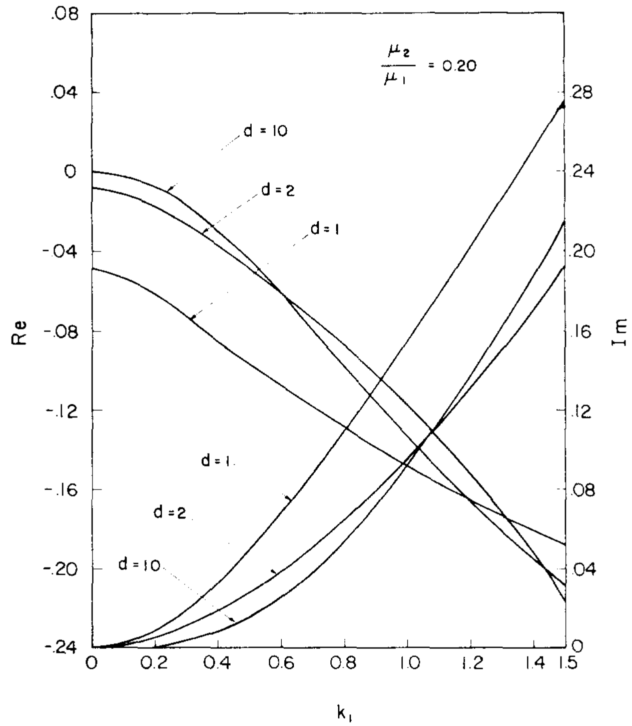

Fig. 3. Dynamic stiffnesses for $\delta = 0{\cdot}20$.

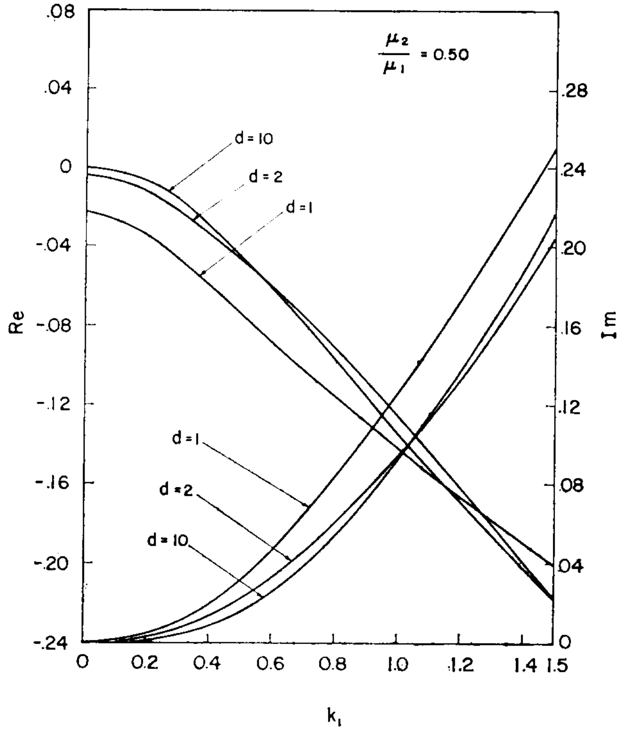

Fig. 4. Dynamic stiffnesses for $\delta = 0{\cdot}50$.

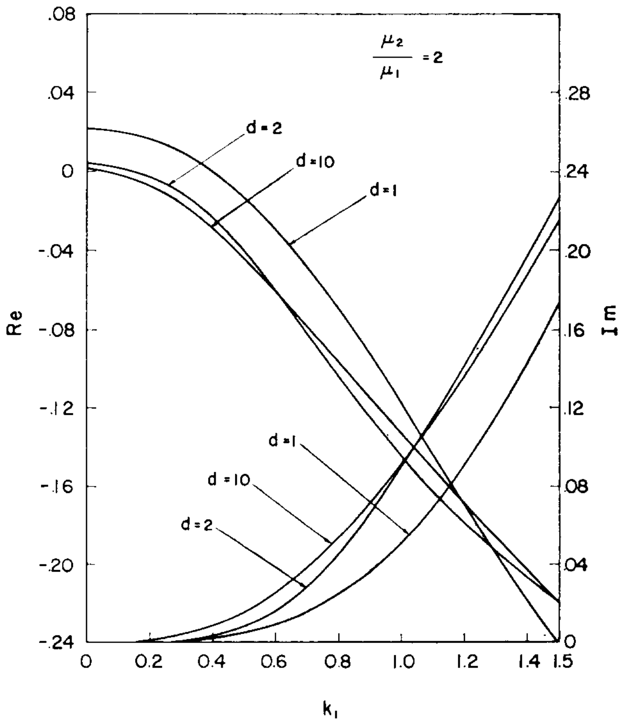

Fig. 5. Dynamic stiffnesses for $\delta = 2{\cdot}0$.

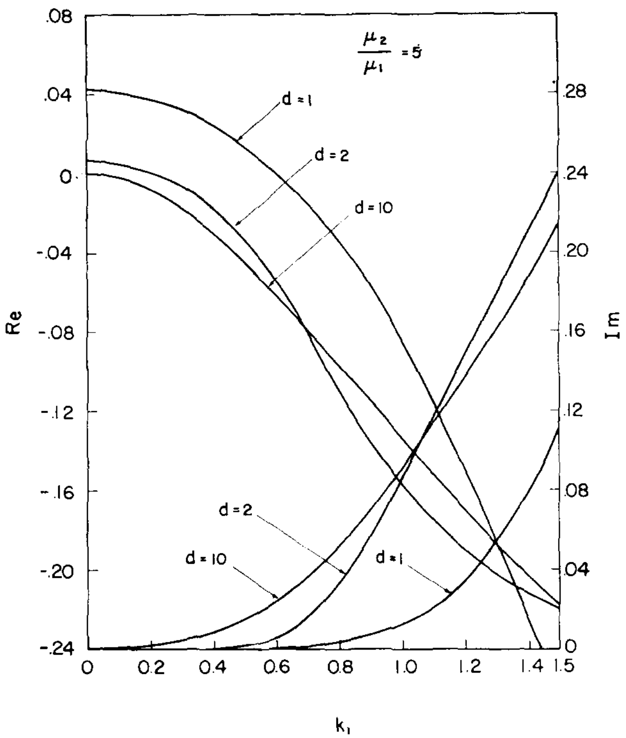

Fig. 6. Dynamic stiffnesses for $\delta = 5{\cdot}0$.

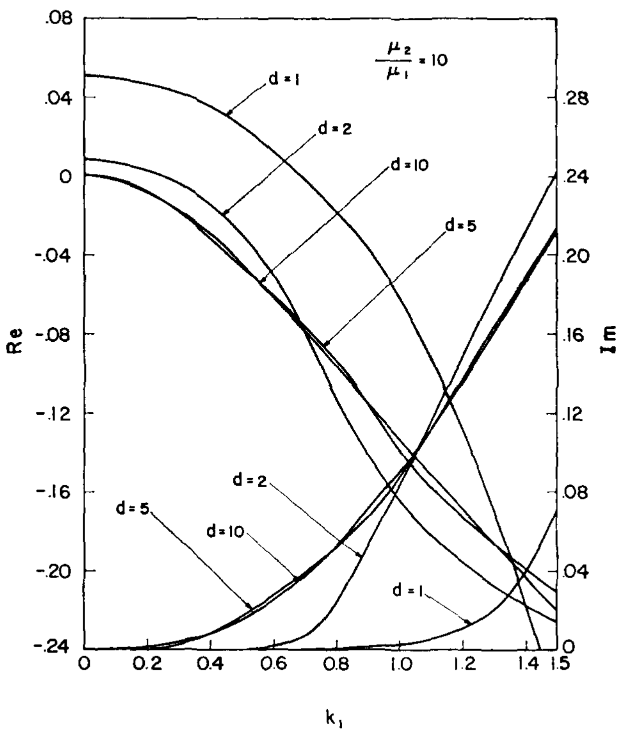

Fig. 7. Dynamic stiffnesses for $\delta = 10{\cdot}0$.

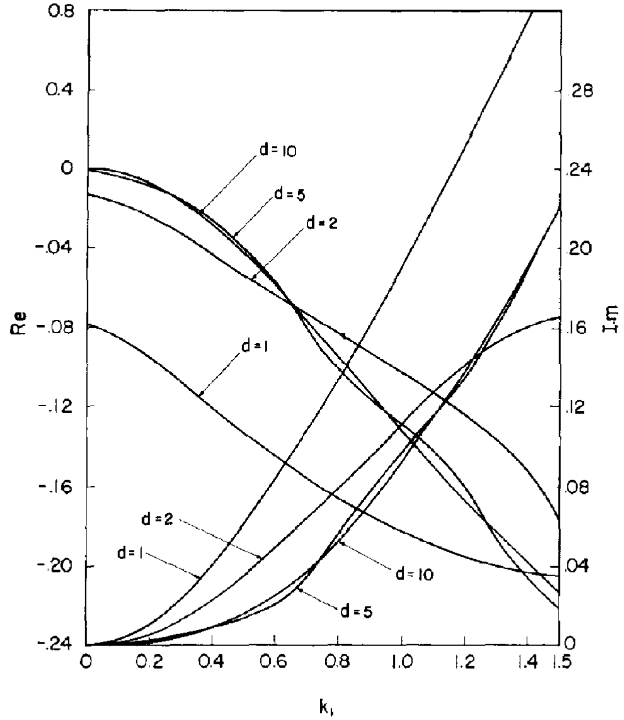

Fig. 8. Dynamic stiffnesses for free boundary ($\delta = 0$).

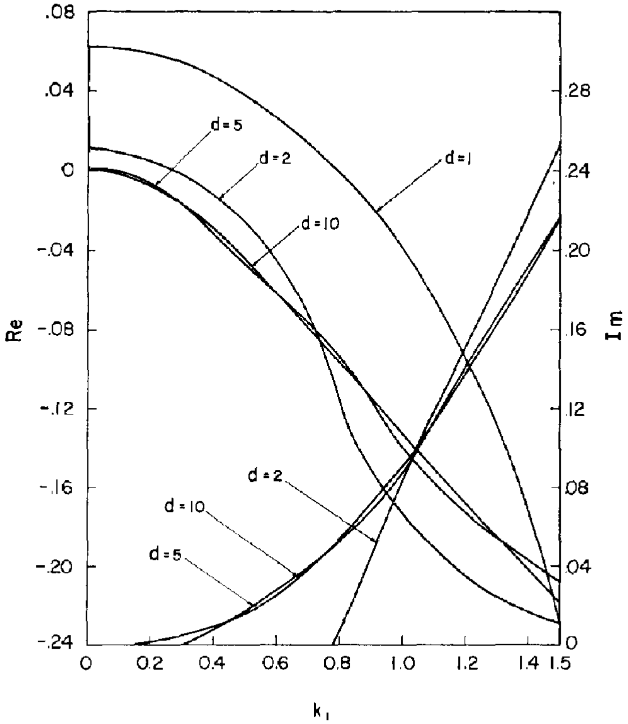

Fig. 9. Dynamic stiffnesses for fixed boundary ($\delta = \infty$).

there is no radiation damping for certain ranges of frequency and a particular geometry. For the case of the layer on the half-space there will be no frequency for which the damping is zero. However, as the half-space becomes very rigid, as in Fig. 7, it is seen that the damping is nearly zero for a range of frequency below the cut-off point corresponding to the rigid foundation case. For comparison purposes the results for a free boundary and a fixed boundary are shown plotted respectively in Figs. 8 and 9. From these figures (which are also given in a more limited form in[3]), one can see how the results for the layer on a half-space go through a smooth transition to these two limiting forms.

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Абстракт — Посредством метода интегрального преобразования, решается задача слоя, соединенного с упругим полупространством. Этот слой находится под влиянием крутильных колебаний от присоединенного жесткого, круглого диска. Применяя общепринятую методику расчета, задача сводится к интегральному уравнению Фредгольма второго рода, ядро которого вызывает подсчитания интегралов главного значения. Определяются динамические жесткости для предела толщин слоя, свойств материала и частот.